PERMANENT AXES OF ROTATION OF A RIGID BODY WITH A FIXED POINT WHEN THE INTEGRALS OF D.N. GORIACHEV EXIST

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Permanent axes of rotation of a rigid body, with one fixed point, in a uniform force field were discovered by Mlodzevskii [1], and Staude [2] in 1894. The cone of the permanent rotations of a heavy rigid body was considered by Rumiantsev [3], who also investigated the stability of these motions.

In the works of Goriachev [4,5] there are given integrals of motion, and the forces acting on the body with a fixed point.

In this work there are determined permanent axes of rotation of a rigid body under the action of forces for which the integrals of Goriachev exist.

Equation of motion. The position of a rigid body with a fixed point O will be described by a rectangular coordinate system $Ox_1x_2x_3$, which is attached to the body in such a way that the coordinate axes coincide with the principal axes of inertia of the body about the fixed point relative to a rectangular coordinate system $O\xi\eta\zeta$, which is fixed in space. The position of the x_i-axis in the fixed system will be determined by nine cosines α_i, β_i, γ_i (i = 1, 2, 3) which are connected by the relations

$$\alpha_i = \beta_{i+1}\gamma_{i+2} - \beta_{i+2}\gamma_{i+1} \qquad (\beta_i, \gamma_i)$$

where each subscript must not exceed 3; this can be accomplished by taking the subscript modulus 3, i.e. subtracting 3 if they exceed 3.

The equations for the direction cosines of the moving axes are

$$d\alpha_i / dt = \alpha_{i+1}p_{i+2} - \alpha_{i+2}p_{i+1} \tag{0.1}$$

Here p_i is the projection of the angular velocity of the rotating body upon the moving axes. The nine equations (0.1) are supplemented by three dynamic equations of Euler:

$$A_i dp_i / dt = (A_{i+1} - A_{i+2})\, p_{i+1}p_{i+2} + L_i \tag{0.2}$$

Here L_i is the moment of the external forces about the axis x_i; A_i is the principal moment of inertia about the same axis.

If the external forces depend only on the position of the rigid body, and if they furthermore, admit a force function U, then the quantities L_i can be expressed in terms of U in the following way:

$$L_i = f(\alpha_i) + f(\beta_i) + f(\gamma_i), \qquad f(x_i) = x_{i+2}\, \partial U / \partial x_{i+1} - x_{i+1}\, \partial U / \partial x_{i+2} \tag{0.3}$$

1. Attraction of several points of a rigid body by a fixed plane. The fourth algebraic integral of the equations (0.1) and (0.2) exists [5] if the moments of inertia $A_1 = A_2 = 2A_3$ and the external forces admit the existence of a force function

$$U = A_1 [a(n-1)^{-1}\gamma_3^{1-n} + {}^1/_2 b(\gamma_2^2 - \gamma_1^2) - c_1\gamma_1 - c_2\gamma_2]\ (n=3) \tag{1.1}$$

Here, we shall assume that $a > b > 0$, $c_k > 0$, and n is a positive integer. If in (1.1) we set $a = b = 0$, then we obtain the case considered by S.V. Kovalevskaia: a constant force is acting along the ζ-axis on a point which lies in the equatorial plane of the body's ellipsoid of inertia. If a and b differ from zero, we have a generalization of the case of Kovalevskaia in the sense that external forces have been added. The term which contains a reveals the action of the force applied to a point on the x_3-axis and acting in the direction of the ζ-axis; the magnitude of this force is inversely proportional to n, the degree of the distance of the point of application from the fixed plane $\xi\eta$. The term that contains b corresponds to the forces R_1 and R_2 which are parallel to the ζ-axis; the force R_1 is applied to a point on the x_1-axis, and the force R_2 to a point of the x_2-axis. The distances of these points from the fixed $\xi\eta$-plane are d_1 and d_2, respectively. Hence, $R_1 = -md_1$, $R_2 = md_2$ where m is a constant.

Let us examine under what conditions the considered body will have a fixed (permanent) axis of rotation. It is known that if the axis of rotation is fixed in space, then it is also fixed in the body. Hence, denoting the angular velocity by ω, the cosines of the angles between the fixed axis and the axes of inertia of the body by l_i, we see that

$p_i = l_i\omega$. The cosines l_i are independent of time, and satisfy the condition

$$l_1^2 + l_2^2 + l_3^2 = 1 \tag{1.2}$$

Computing the external moments by means of formulas (0.3), we obtain the equations of motion of the rigid body

$$\begin{aligned} l_1 d\omega / dt &= {}^1/_2 l_2 l_3 \omega^2 + b\gamma_2\gamma_3 + a\gamma_2\gamma_3^{-n} - c_2\gamma_3 \\ l_2 d\omega / dt &= -{}^1/_2 l_1 l_3 \omega^2 + b\gamma_1\gamma_3 - a\gamma_1\gamma_3^{-n} + c_1\gamma_3 \end{aligned} \tag{1.3}$$

$${}^1/_2\, l_3\, d\omega / dt = c_2\gamma_1 - c_1\gamma_2 - 2b\gamma_1\gamma_2, \qquad d\gamma_i / dt = \omega\,(l_{i+2}\gamma_{i+1} - l_{i+1}\gamma_{i+2}) \tag{1.4}$$

Equations (1.3) yield the integrals of the area and of the kinetic energy

$$(l_1\gamma_1 + l_2\gamma_2 + {}^1/_2\, l_3\gamma_3)\,\omega = k_1 \tag{1.5}$$

$$(l_1^2 + l_2^2 + {}^1/_2\, l_3^2)\,\omega^2 = 2\,[a\,(n-1)^{-1}\gamma_3^{1-n} + {}^1/_2\, b\,(\gamma_2^2 - \gamma_1^2) - c_1\gamma_1 - c_2\gamma_2 + h] = 2n$$

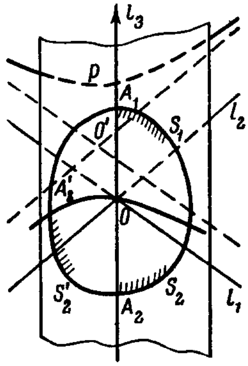
By eliminating the angular velocity from these equations, we derive

$$l_1\gamma_1 + l_2\gamma_2 + {}^1/_2\, l_3\gamma_3 = k_1\,(2n)^{-1/2}\sqrt{l_1^2 + l_2^2 + {}^1/_2\, l_3^2} \tag{1.6}$$

Equations (1.4) have the solution

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \qquad l_1\gamma_1 + l_2\gamma_2 + l_3\gamma_3 = k_2 \tag{1.7}$$

1) If the relation (1.6) is not a consequence of the second equation in (1.7) then the system of equations (1.6) and (1.7) can be used for the determination of the γ_i. If this system is solvable for the γ_i, then these quantities will be expressed in terms of k_1, k_2, l_i, a, b, c_k and h, and they will, therefore, be constants. But then equations (1.4) will yield



$$\gamma_1 / l_1 = \gamma_2 / l_2 = \gamma_3 / l_3 \tag{1.8}$$

From this, and from (1.3) and (1.7), it now follows that

$$l_1 = \gamma_1, \quad l_2 = \gamma_2, \quad l_3 = \gamma_3 \tag{1.9}$$

i.e. the permanent axis will be the ζ-axis. The angular velocity, which can be determined by means of the integrals of the kinetic energy, will hereby also be constant. The equations that determine the permanent axes in the rigid body will have the form

$$[(-1)^i b - {}^1/_2\,\omega^2]\, l_{i+1} l_{i+2} - a l_{3-i} l_3^{-n} + c_{3-i} l_3 = 0$$
$$(i = 1, 2) \qquad (1.10)$$
$$c_2 l_1 - c_1 l_2 - 2b l_1 l_2 = 0 \qquad (1.11)$$

Elimination of ω^2 from the equations in (1.10) leads to equation (1.11). This equation determines in the $Ol_1 l_2 l_3$ space a hyperbolic cylinder with generators parallel to Ol_3 and passing through a rectangular hyperbola, which lies (see figure) in the plane $l_1 l_2$, and whose vertex is at the point O' $(-c_1/2b,\ c_2/2b)$; its semi-axis is equal to $1/b\sqrt{(c_1 c_2/2)}$, and the real axis is parallel to the bisector of the second and fourth quadrants.

The values of the direction cosines l_i satisfy, in addition, condition (1.2). Hence, if one draws a unit sphere with center at the fixed point, then the locus of the points of intersection of the hyperbolic cylinder (1.11) with this sphere will consist of two closed branches of some curve (lying on the sphere) to each point of which there corresponds one of the permanent axes, and, conversely, to every permanent axis there corresponds one point of this curve on the sphere. One of the branches of the cylinder (1.11) will pass through the l_3-axis. Therefore, one branch of the curve on the sphere will always exist. It passes along the cylinder and sphere (see figure) through the points $A_1(0,\ 0,\ 1)$ and $A_2(0,\ 0,\ -1)$. The second branch of the curve on the sphere can exist when the coordinates of the point $P(x_1,\ x_2,\ 0)$ nearest to the origin satisfy the condition $x_1^2 + x_2^2 \leqslant 1$, where the values of x_i are found from equation (1.11) and from the condition that the normal to the hyperbola (1.11)

$$(l_1 - x_1)(c_1 + 2bx_1) + (l_2 - x_2)(c_2 - 2bx_2) = 0$$

should pass through the origin of the coordinate system. This is equivalent to requiring that

$$c_1 x_1 + c_2 x_2 + 2b\,(x_1^2 - x_2^2) = 0 \qquad (1.12)$$

It should be mentioned that only those axes can be permanent axes for which equations (1.10) yield a positive value for ω^2. Following Staude, we shall call all such lines, and also all the points of the curve on the sphere that correspond to these lines, admissible lines or points for this problem; all other lines and points will be called inadmissible lines or points, respectively.

In the given case, the admissible points will be the points of the curve on the sphere, the coordinates of which satisfy the conditions

$$c_i / l_i \geqslant a l_3^{-n-1} + (-1)^i b \qquad (i = 1, 2) \tag{1.13}$$

Let us consider the cylindrical surfaces

$$l_i = \frac{c_i l_3^{n+1}}{a + (-1)^i b l_3^{n+1}} \qquad (i = 1, 2) \tag{1.14}$$

If n is odd, then the surfaces (1.14) cut the first curve on the sphere (see figure) at the points S_1 and S_2 which are located symmetrically with respect to the plane $l_1 l_2$. All points on the axes $A_1 S_2$ and $A_2 S_2$ will be admissible points. If n is even, then the surfaces (1.4) will cut the first curve on the sphere at the points S_1 and S_2'. All points lying on the arcs $A_1 S_1$ and $A_2 S_2'$ will satisfy conditions (1.13).

There will be no admissible points on the second curve on the sphere.

2) If equations (1.6) are implied by (1.7), then γ_i and ω need not be constants. In this case the permanent axes will be of a type different from those considered above. The equivalence of (1.6) to (1.7) implies that the coefficients of the γ_i in these equations are proportional, i.e.

$$\frac{l_1}{l_1} = \frac{l_2}{l_2} = \frac{l_3}{2l_3} = \frac{k_1 \sqrt{l_1^2 + l_2^2 + l_3^2}\,{}^{1/2}}{\sqrt{2n}k_2} \tag{1.15}$$

Equations (1.15) will be satisfied when $l_3 = 1$, $k_1 = k_2 = l_1 = l_2 = 0$. Then the area integrals imply that $\gamma_3 = 0$. These properties are possessed by a rigid body which rotates around an x_3-axis which lies in the plane that attracts the body. Without loss of generality, we may line up x_3 with ξ. In the plane $x_1 x_2$ which is orthogonal to the axis of rotation there lie three forces with constant directions; one of these forces is of constant magnitude (the force of gravity), while the absolute values of the remaining forces are proportional to the distances of the points of their application from the η-axis. We thus have a physical pendulum with two auxiliary forces. Another possibility for the validity of (1.15) is that $l_1 = l_2$ and $l_3 = 0$. But if two moments of inertia are equal, we may change the directions of the axes so that one of the cosines that corresponds to the equal axes becomes zero. Hence, without loss of generality, we may set $l_1 = 1$, $l_2 = l_3 = 0$. Besides that, the conditions $k_1 = k_2 = 0$ must also be fulfilled. From the area integral it then follows that $\gamma_1 = 0$, i.e. the 0ζ-axis lies in the plane $x_1 x_2$. This corresponds to a rotation of the body around the x_1-axis which lies in the plane that attracts the body (on the ξ-axis) under the action of three forces of constant direction; one of these forces is constant in absolute value, while the absolute values of the remaining forces are equal to $m_1\delta_1^{-n}$ and $m_2\delta_2$, where the m_i are constants, while the δ_i are the distances of the points of application of the forces from the η-axis. This case also is analogous to a physical pendulum with two new forces.

2. Attraction of a point on the axis of symmetry of the body by a fixed plane. Goriachev [5] has obtained a new algebraic integral for the motion of a rigid body for which $A_1 = A_2$, and the external forces admit a force function

$$U = aA_1(n-1)^{-1}\gamma_3^{1-n} \tag{2.1}$$

where a is a positive constant, and $n = 3$. We shall consider the case when n has any value. Function (2.1) implies the presence of a force, applied to a point of the x_3-axis, and directed along the ζ-axis, while its magnitude is inversely proportional to n, the degree of the distance of the point of application of the force from the $\xi\eta$-plane. The dynamical equations of motion (0.2) have the form

$$(-1)^i l_i\, d\omega / dt = (\varepsilon - 1)\, l_{i+1}\, l_{i+2} - a\gamma_{3-i}\gamma_3^{-n} \quad (i = 1, 2), \qquad l_3 d\omega / dt = 0$$
$$(\varepsilon = A_3 / A_1) \tag{2.2}$$

which must be supplemented by equations (1.4). Equations (2.2) imply that a permanent axis with a variable angular velocity must be a straight line which lies in the equatorial plane of the ellipsoid of inertia $l_3 = 0$. From this we conclude that $l_1\gamma_1 + l_2\gamma_2 = 0$. Without loss of generality, we may set $l_2 = 0$. Then l_1 must be zero. This corresponds to a rotation of the body around the x_1-axis, which coincides with the ξ-axis, the action force is hereby located in the x_2x_3-plane.

The permanent axes with constant angular velocities are determined by the equations

$$\gamma_3^{n-1} = \frac{2a}{(n-1)\,[h - (l_1^2 + l_2^2 + \varepsilon l_3^2)]}, \qquad l_1\gamma_2 - l_2\gamma_1 = 0 \tag{2.3}$$

The first one of these is the integral of the kinetic energy; the second one is a consequence of (2.2). From (1.4) and (2.3) it follows that the permanent axis coincides with the ζ-axis, i.e. equation (1.9) is satisfied. The admissible points on the sphere (1.2) are determined by the relation

$$\omega^2 = \frac{a}{(\varepsilon - 1)\, l_3^{n+1}} > 0$$

which will be satisfied when n is even, on the hemisphere $l_3 < 0$ if $\varepsilon < 1$, and on the hemisphere $l_3 > 0$ if $\varepsilon > 1$; when n is odd, it will be satisfied on both hemispheres if $\varepsilon > 1$, and not be satisfied at all if $\varepsilon < 1$.

3. Action of a force which is constant with respect to the body. The integral p_3 = const will exist also (in addition to considered cases) when the symmetric body $A_1 = A_2$ is subjected to the action of a force FA_1 constant in magnitude, parallel to the x_3-axis, and applied to an arbitrary point x_{i0} of the body. The dynamic equations

$$(-1)^i l_i d\omega / dt = (\varepsilon - 1) l_{i+1} l_{i+2}\omega^2 - Fx_{3-i,0} \quad l_3 d\omega / dt = 0 \qquad (\varepsilon = A_3 / A_1) \tag{3.1}$$

show that the permanent axes with variable angular velocity exist if

$$l_3 = l_1 x_{10} + l_2 x_{20} = 0 \tag{3.2}$$

i.e. the permanent axis is orthogonal to the straight line which passes through the point of application of the force and through the origin, and lies in the x_1x_2-plane. The x_1-axis has this property. If, however, the angular velocity is constant, then equations (3.1) yield

$$x_{10} / l_1 = x_{20} / l_2 = k \tag{3.3}$$

i.e. the permanent axis lies in the same plane with the straight line that passes through the point of application of the force and through the origin of the coordinate system. The acting force lies in the same plane. The angular velocity in this case is given by

$$\omega^2 = \frac{kF}{l_3 (\varepsilon - 1)} \tag{3.4}$$

If follows from this that if $k > 0$, the admissible points of the sphere (1.2) will be points of the hemisphere $l_3 < 0$ if $\varepsilon < 1$, of the hemisphere $l_3 > 0$ if $\varepsilon > 1$.

4. Attraction of a body by a fixed plane. Goriachev's method for finding the integrals of motion yields a positive result also when the force function has the form

$$U = -\alpha (A_1\gamma_1^2 + A_2\gamma_2^2 + A_3\gamma_3^2) \tag{4.1}$$

This, with $\alpha > 0$, corresponds to the attraction of a body by the $\xi\eta$-plane with forces that are proportional to the distances of the points of the body from the plane. This problem has been investigated by de Brun [6]. In studying the stability of a body in the force field (4.1), Beletskii [7] revealed one permanent axis; the constant rotation around the inertia axis perpendicular to the attracting plane. We shall indicate all possible permanent axes of rotation. If we adjoin to equations (1.4) the dynamical equations

$$A_i l_i d\omega / dt = (A_{i+1} - A_{i+2}) (l_{i+1} l_{i+2}\omega^2 - 2\alpha\gamma_{i+1}\gamma_{i+2}) \tag{4.2}$$

we obtain the integrals

$$(A_1l_1^2 + A_2l_2^2 + A_3l_3^2)\omega^2 = h - 2\alpha(A_1\gamma_1^2 + A_2\gamma_2^2 + A_3\gamma_3^2)$$
$$(A_1l_1\gamma_1 + A_2l_2\gamma_2 + A_3l_3\gamma_3)\omega = k_1$$

which yield

$$A_1l_1\gamma_1 + A_2l_2\gamma_2 + A_3l_3\gamma_3 = k_1\left(\frac{A_1l_1^2 + A_2l_2^2 + A_3l_3^2}{h - 2\alpha(A_1\gamma_1^2 + A_2\gamma_2^2 + A_3\gamma_3^2)}\right)^{1/2} \tag{4.3}$$

1) If (4.3) and (1.7) are independent of each other, then the system of equations (1.7) and (4.3) determine γ_i as constants. But then, just as above, we obtain relation (1.9), i.e. every permanent axis must be perpenduclar to the attracting plane. From the derived integrals it follows that ω = constant. Equations (4.2) now take on the form

$$(A_{i+1} - A_{i+2})(\omega^2 - 2\alpha)\, l_{i+1}l_{i+2} = 0 \tag{4.4}$$

This yields the following results:

a) When $\omega^2 \neq 2\alpha$ and $A_1 \neq A_2 \neq A_3$ (or $A_1 = A_2 \neq A_3$) only the principal axes of the ellipsoid of inertia can be permanent axes.

b) If the ellipsoid of inertia degenerates into a sphere $A_1 = A_2 = A_3$, or if $\omega^2 = 2\alpha$, then there can be a rotation of the body around a line that is perpendicular to the plane; the position of the body relative to this line is immaterial.

2) Equation (4.3) is implied by (1.7) if $k_1 = k_2 = 0$.

Then the quantities γ_i and ω need not be constants. The relation between equations (4.3) and (1.7) is the following:

$$\frac{A_1l_1}{l_1} = \frac{A_2l_2}{l_2} = \frac{A_3l_3}{l_3} \tag{4.5}$$

Condition (4.5) can be met if one equates all three moments of inertia, or equates two moments of inertia and sets the cosine that corresponds to the distinct moment, equal to zero, or if one makes two of the cosines l_i equal to zero. When there are two or three equal moments of inertia, one can change the direction of the axes in such a way that one of the cosines corresponding to the equal axes becomes zero. Without loss of generality, we may set $l_1 = 1$, $l_2 = l_3 = 0$. Then the second equation of (1.7) or (4.3) implies that $\gamma_1 = 0$. This corresponds to a rotation of the body around one of the axes of inertia (x_1) which lies in the attracting plane $\xi\eta$. An analogous result has been obtained by Beletskii [8] for the motion of a body in a central force field.

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